## AN EXPLANATION OF JOINT DIAGRAMS

When bolted joints are subjected to external tensile loads, what forces and elastic deformation really exist? The majority of engineers in both the fastener manufacturing and user industries still are uncertain. Several papers, articles, and books, reflecting various stages of research into the problem have been published and the volume of this material is one reason for confusion. The purpose of this article is to clarify the various explanations that have been offered and to state the fundamental concepts which apply to forces and elastic deformations in concentrically loaded joints. The article concludes with general design formulae that take into account variations in tightening, preload loss during service, and the relation between preloads, external loads and bolt loads.

### The Joint Diagram

Forces less than proof load cause elastic strains. Conversely, changes in elastic strains produce force variations. For bolted joints this concept is usually demonstrated by joint diagrams.

The most important deformations within a joint are elastic bolt elongation and elastic joint compression in the axial direction. If the bolted joint in Fig. 1 is subjected to the preload $F_i$ the bolt elongates as shown by the line OB in Fig. 2A and the joint compresses as shown by the line OJ. These two lines, representing the spring characteristics of the bolt and joint, are combined into one diagram in Fig. 2B to show total elastic deformation.

If a concentric external load $F_e$ is applied under the bolt head and nut in Fig. 1, the bolt elongates an additional amount while the compressed joint members partially relax. These changes in deformation with external loading are the key to the interaction of forces in bolted joints.

In Fig. 3A the external load $F_e$ is added to the joint diagram Fe is located on the diagram by applying the upper end to an extension of OB and moving it in until the lower end contacts OJ. Since the total amount of elastic deformation (bolt plus joint) remains constant for a given preload, the external load changes the total bolt elongation to $\Delta l_B + \lambda$ and the total joint compression to $\Delta l_J - \lambda$.

In Fig. 3B the external load $F_e$ is divided into an additional bolt load $F_{eB}$ and the joint load $F_{eJ}$, which unloads the compressed joint members. The maximum bolt load is the sum of the load preload and the additional bolt load:

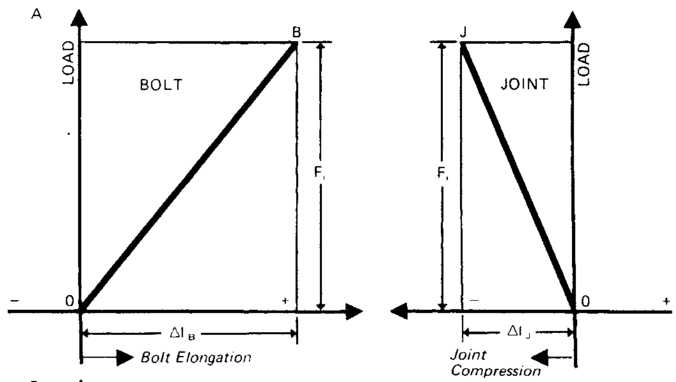
$$F_{B\,max} = F_i + F_{eB}$$

If the external load Fe is an alternating load, $F_{eB}$ is that part of $F_e$ working as an alternating bolt load, as shown in Fig. 3B. This joint diagram also illustrates that the joint absorbs more of the external load than the bolt subjected to an alternating external load.

The importance of adequate preload is shown in Fig. 3C. Comparing Fig. 3B and Fig. 3C, it can be seen that $F_{eB}$ will remain relatively small as long as the preload $F_i$ is greater than $F_{eJ}$. Fig. 3C represents a joint with insufficient preload. Under this condition, the amount of external load that the joint can absorb is limited, and the excess load must then be applied to the bolt. If the external load is alternating, the increased stress levels on the bolt produce a greatly shortened fatigue life.

When seating requires a certain minimum force or when transverse loads are to be transformed by friction, the minimum clamping load $F_{J\,min}$ is important.

$$F_{J\,min} = F_{B\,max} - F_e$$



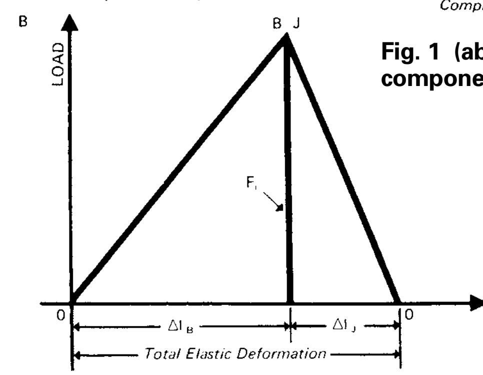




**Fig. 1 (above) Joint components**

**Fig. 2 Joint diagram is obtained by combining** load vs. deformation diagrams of bolt and joints.

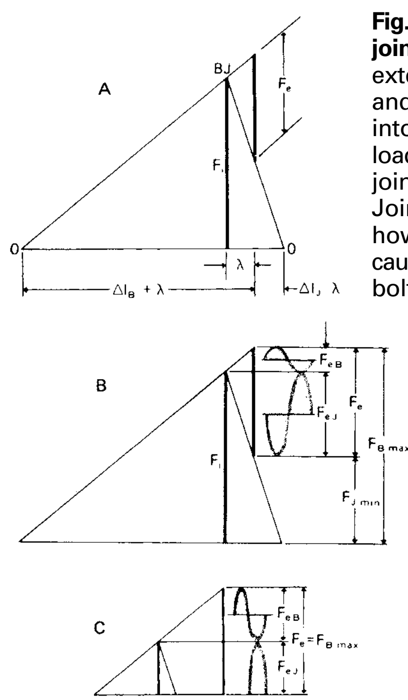



**Fig. 3 The complete simple joint diagrams** show external load $F_e$ added (A), and external load divided into an additional bolt load $F_{eB}$ and reduction in joint compression $F_{eJ}$ (B). Joint diagram (C) shows how insufficient preload $F_i$ causes excessive additional bolt load $F_{eB}$.

## Spring Constants

To construct a joint diagram, it is necessary to determine the spring rates of both bolt and joint. In general, spring rate is defined as:

$$K = \frac{F}{\Delta l}$$

From Hook's law:

$$\Delta l = \frac{lF}{EA}$$

Therefore:

$$K = \frac{EA}{l}$$

To calculate the spring rate of bolts with different cross sections, the reciprocal spring rates, or compliances, of each section are added:

$$\frac{1}{K_B} = \frac{1}{K_1} + \frac{1}{K_2} + \ \cdots\ + \frac{1}{K_n}$$

Thus, for the bolt shown in Fig. 4:

$$\frac{1}{K_B} = \frac{1}{E}\left(\frac{0.4d}{A_1} + \frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_m} + \frac{0.4d}{A_m}\right)$$

where

d = the minor thread diameter and

$A_m$ = the area of the minor thread diameter

This formula considers the elastic deformation of the head and the engaged thread with a length of 0.4d each.

Calculation of the spring rate of the compressed joint members is more difficult because it is not always obvious which parts of the joint are deformed and which are not. In general, the spring rate of a clamped part is:

$$K_J = \frac{EA_S}{l_J}$$

where $A_s$ is the area of a substitute cylinder to be determined.

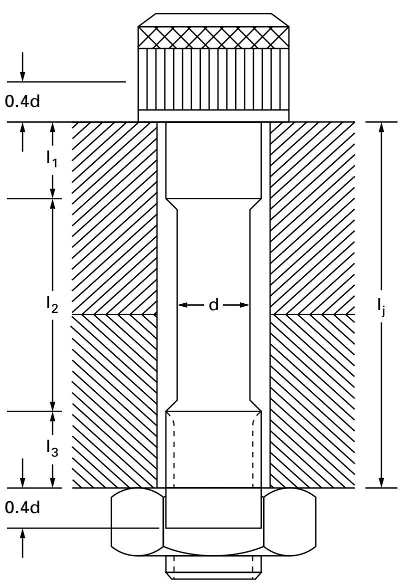



**Fig. 4 Analysis of bolt lengths** contributing to the bolt spring rate.

When the outside diameter of the joint is smaller than or equal to the bolt head diameter, i.e.,as in a thin bushing, the normal cross sectioned area is computed:

$$A_s = \frac{\pi}{4}(D_c^2 - D_h^2)$$

where

$D_c$ = OD of cylinder or bushing and

$D_h$ = hole diameter

When the outside diameter of the joint is larger than head or washer diameter $D_H$, the stress distribution is in the shape of a barrel, Fig 5. A series of investigations proved that the areas of the following substitute cylinders are close approximations for calculating the spring contents of concentrically loaded joints.

When the joint diameter $D_J$ is greater than $D_H$ but less than $3D_H$;

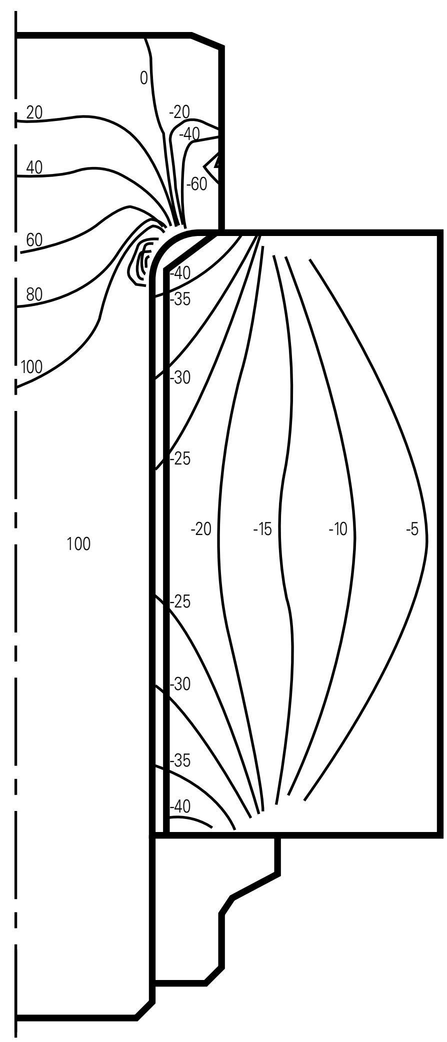



**Fig. 5 Lines of equal axial stresses in a bolted joint** obtained by the axisymmetric finite element method are shown for a 9/16–18 bolt preloaded to 100 KSI. Positive numbers are tensile stresses in KSI; negative numbers are compressive stresses in KSI.

$$A_s = \frac{\pi}{4}(D_H^2 - D_h^2) + \frac{\pi}{8}\left(\frac{D_J}{DH} - 1\right)\left(\frac{D_H l_J}{5} + \frac{l_J^2}{100}\right)$$

When the joint diameter $D_J$ is equal to or greater than $3D_H$:

$$A_s = \frac{\pi}{4}[(D_H + 0.1\, l_J)^2 - D_h^2]$$

These formulae have been verified in laboratories by finite element method and by experiments.

Fig. 6 shows joint diagrams for springy bolt and stiff joint and for a stiff bolt and springy joint. These diagrams demonstrate the desirability of designing with springy bolt and a stiff joint to obtain a low additional bolt load $F_{eB}$ and thus a low alternating stress.

### The Force Ratio

Due to the geometry of the joint diagram, Fig. 7,

$$F_{eB} = \frac{F_e K_B}{K_B + K_J}$$

Defining $\Phi = \dfrac{K_B}{K_B + K_J}$

$F_{eB} = F_e\Phi$ and

$\Phi$, called the Force Ratio, $= \dfrac{F_{eB}}{F_e}$

For complete derivation of $\Phi$, see Fig. 7.

To assure adequate fatigue strength of the selected fastener the fatigue stress amplitude of the bolt resulting from an external load $F_e$ is computed as follows:

$$\sigma_B = \pm \frac{F_{eB}/2}{A_m} \quad \text{or}$$

$$\sigma_B = \pm \frac{\Phi F_e}{2 A_m}$$

### Effect of Loading Planes

The joint diagram in Fig 3, 6 and 7 is applicable only when the external load $F_e$ is applied at the same loading planes as the preloaded $F_i$, under the bolt head and the nut. However, this is a rare case, because the external load usually affects the joint somewhere between the center of the joint and the head and the nut.

When a preloaded joint is subjected to an external load $F_e$ at loading planes 2 and 3 in Fig. 8, $F_e$ relieves the compression load of the joint parts between planes 2 and 3. The remainder of the system, the bolt and the joint parts between planes 1-2 and 3-4, feel additional load due to $F_e$ applied planes 2 and 3, the joint material between planes 2 and 3 is the clamped part and all other joint members, fastener and remaining joint material, are clamping parts. Because of the location of the loading planes, the joint diagram changes from black line to the blue line. Consequently, both the additional bolt load $F_{B\,max}$ decrease significantly when the loading planes of $F_e$ shift from under the bolt head and nut toward the joint center.

Determination of the length of the clamped parts is, however, not that simple. First, it is assumed that the external load is applied at a plane perpendicular to the bolt axis. Second, the distance of the loading planes from each other has to be estimated. This distance may be expressed as the ratio of the length of clamped parts to the total joint length. Fig. 9 shows the effect of two different loading planes on the bolt load, both joints having the same preload $F_i$ and the same external load $F_e$. The lengths of the clamped parts are estimated to be $0.75 l_J$ for joint A, and $0.25 l_J$ for joint B.

In general, the external bolt load is somewhere between $F_{eB} = 1\Phi F_e$ for loading planes under head and nut and $F_{eB} = 0\Phi F_e = 0$ when loading planes are in the joint center, as shown in Fig. 10. To consider the loading planes in calculations, the formula:

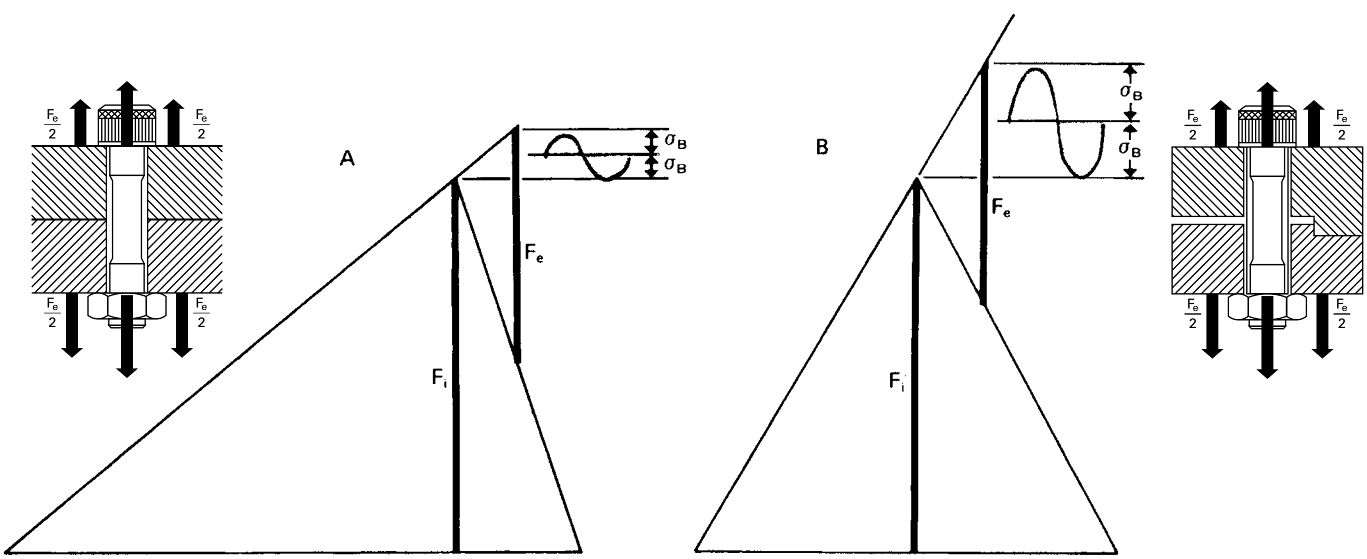



**Fig. 6 Joint diagram of a springy bolt in a stiff joint (A), is compared** to a diagram of a stiff bolt in a springy joint (B). Preload $F_i$ and external load $F_e$ are the same but diagrams show that alternating bolt stresses are significantly lower with a spring bolt in a stiff joint.

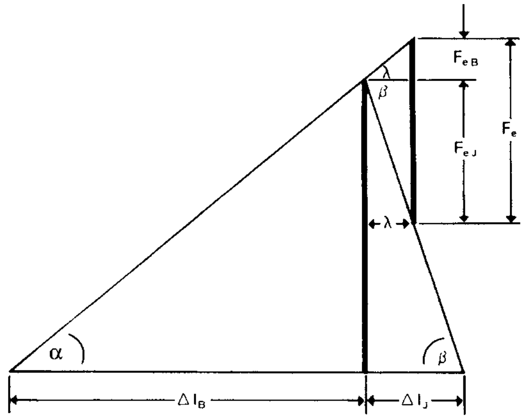



**Fig. 7 Analysis of external load $F_e$** and derivation of Force Ratio Φ.

$$\tan\alpha = \frac{F_i}{\Delta l_B} = K_B \text{ and } \tan\beta = \frac{Fi}{\Delta l_J} = K_J$$

$$\lambda = \frac{F_{eB}}{\tan\alpha} = \frac{F_{eJ}}{\tan\beta} = \frac{FeB}{K_B} = \frac{F_{eJ}}{K_J} \quad \text{or}$$

$$F_{eJ} = \lambda \tan\beta \text{ and } F_{eB} = \lambda \tan\alpha$$

Since $F_e = F_{eB} + F_{eJ}$
$F_e = F_{eB} + \lambda \tan\beta$

Substituting $\frac{F_{eB}}{\tan\alpha}$ for λ produces:

$$F_e = F_{eB} + \frac{F_{eB}\tan\beta}{\tan\alpha}$$

Multiplying both sides by tan α:

$$F_e \tan\alpha = F_{eB} (\tan\alpha + \tan\beta) \text{ and}$$

$$F_{eB} = \frac{F_e \tan\alpha}{\tan\alpha \tan\beta}$$

Substituting $K_B$ for tan α and $K_J$ for tan β

$$F_{eB} = F_e \frac{F_B}{K_B + K_J}$$

Defining $\Phi = \frac{K_B}{K_B + K_J}$

$$F_{eB} = \Phi F_e$$

$$\Phi = \frac{F_{eB}}{F_e}$$ and it becomes obvious why Φ is called force ratio.

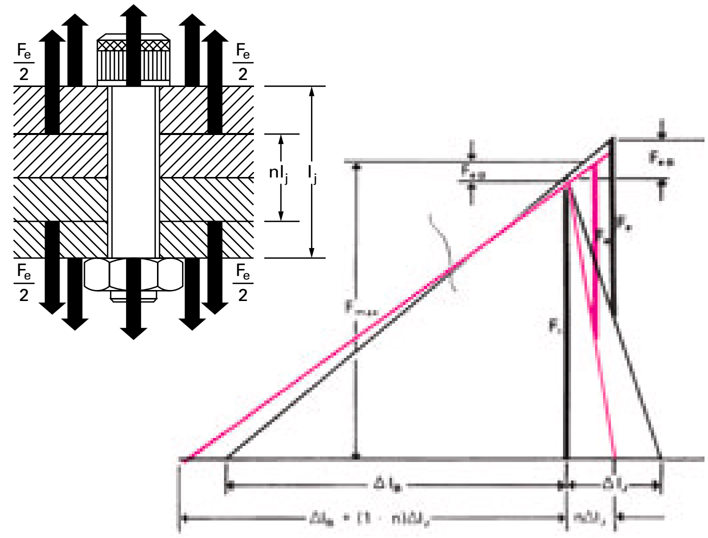



**Fig. 8 Joint diagram shows effect of loading planes of $F_e$** on bolt loads $F_{eB}$ and $F_{B\,max}$. Black diagram shows $F_{eB}$ and $F_{B\,max}$ resulting from $F_e$ applied in planes 1 and 4. Orange diagram shows reduced bolt loads when Fe is applied in planes 2 and 3.

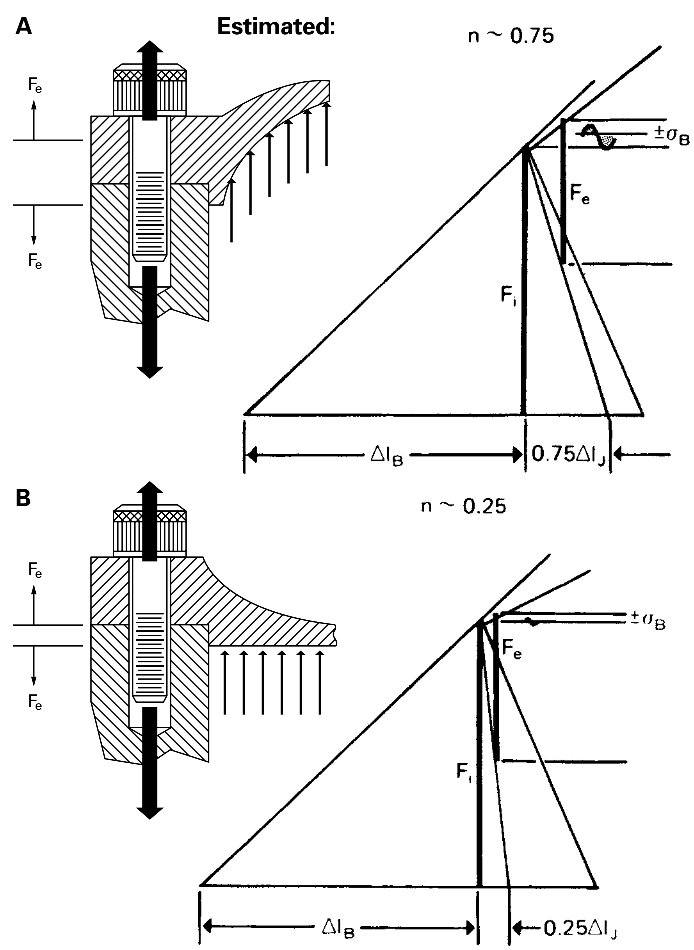



**Fig. 9 When external load is applied relatively near bolt head,** joint diagram shows resulting alternating stress $\alpha_B$ (A). When same value external load is applied relatively near joint center, lower alternating stress results (B).

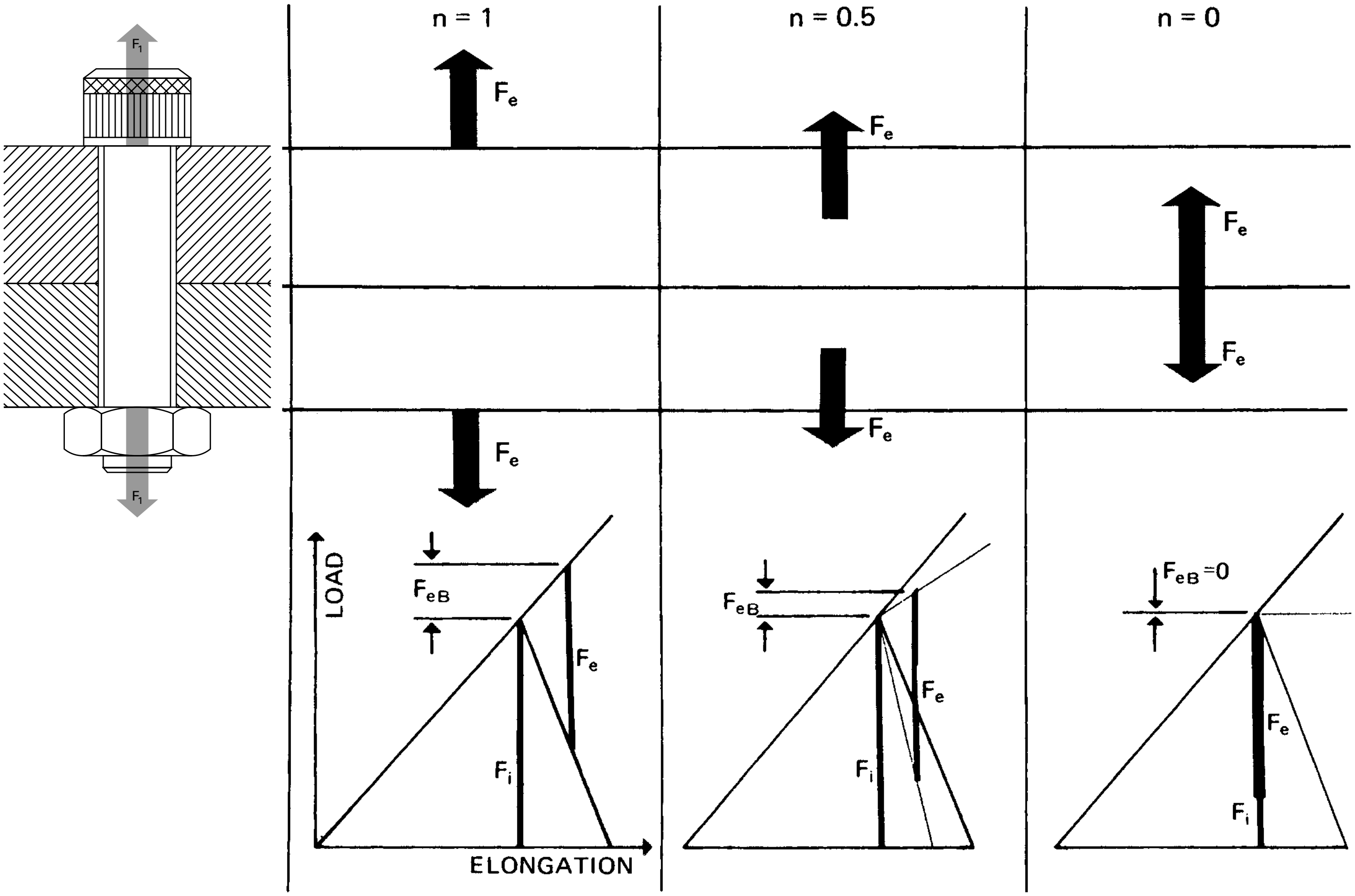



**Fig. 10 Force diagrams show the effect** of the loading planes of the external load on the bolt load.

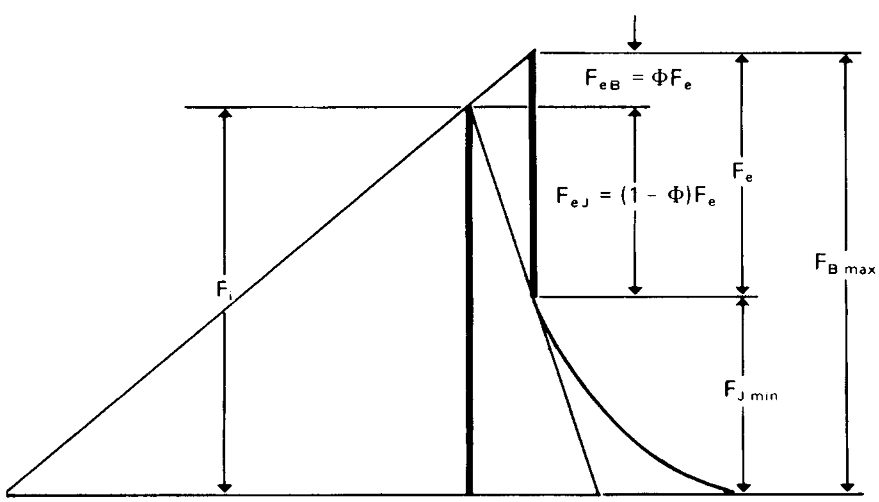



**Fig. 11 Modified joint diagram** shows nonlinear compression of joint at low preloads.

$F_{eB} = \Phi F_e$ must be modified to :

$F_{eB} = n \Phi F_e$

where n equals the ratio of the length of the clamped parts due to $F_e$ to the joint length $l_j$. The value of n can range from 1, when $F_e$ is applied under the head and nut, to O, when $F_e$ is applies at the joint center. Consequently the stress amplitude:

$$\sigma_B = \pm \frac{\Phi F_e}{2 A_m} \quad \text{becomes}$$

$$\sigma_B = \pm \frac{n \Phi F_e}{2 A_m}$$

### General Design Formulae

Hitherto, construction of the joint diagram has assumed linear resilience of both bolt and joint members. However, recent investigations have shown that this assumption is not quite true for compressed parts.

Taking these investigations into account, the joint diagram is modified to Fig. 11. The lower portion of the joint spring rate is nonlinear, and the length of the linear portion depends on the preload level $F_i$. The higher $F_i$ the longer the linear portion. By choosing a sufficiently high minimum load, $F_{min}>2F_e$, the non-linear range of the joint spring rate is avoided and a linear relationship between $F_{eB}$ and $F_e$ is maintained.

Also from Fig. 11 this formula is derived:

$$F_{i\,min} = F_{J\,min} + (1 - \Phi) F_e + \Delta F_i$$

where $\Delta F_i$ is the amount of preload loss to be expected. For a properly designed joint, a preload loss $\Delta F_i = -(0.005 \text{ to } 0.10) F_i$ should be expected.

The fluctuation in bolt load that results from tightening is expressed by the ratio:

$$a = \frac{F_{i\,max}}{F_{i\,min}}$$

where a varies between 1.25 and 3.0 depending on the tightening method.

Considering a the general design formulae are:

$$F_{i\,nom} = F_{J\,min} = (1 - \Phi) F_e$$

$$F_{i\,max} = a\,[\,F_{j\,min} + (1 - \Phi) F_e + \Delta F_i\,]$$

$$F_{B\,max} = a\,[\,F_{j\,min} + (1 - \Phi) F_e + \Delta F_i\,] + \Phi F_e$$

### Conclusion

The three requirements of concentrically loaded joints that must be met for an integral bolted joint are:

1. The maximum bolt load FB max must be less than the bolt yield strength.

2. If the external load is alternating, the alternating stress must be less than the bolt endurance limit to avoid fatigue failures.

3. The joint will not lose any preload due to permanent set or vibration greater than the value assumed for $\Delta F_i$ .

## SYMBOLS

| | |
|---|---|
| A | Area (in.$^2$) |
| Am | Area of minor thread diameter (in.$^2$) |
| $A_s$ | Area of substitute cylinder (in.$^2$) |
| $A_x$ | Area of bolt part $1_x$ (in.$^2$) |
| d | Diameter of minor thread (in.) |
| $D_c$ | Outside diameter of bushing (cylinder) (in.) |
| $D_H$ | Diameter of Bolt head (in.) |
| $D_h$ | Diameter of hole (in.) |
| $D_J$ | Diameter of Joint |
| E | Modulus of Elasticity (psi) |
| F | Load (lb) |
| $F_e$ | External load (lb.) |
| $F_{eB}$ | Additinal Bolt Load due to external load (lb) |
| $F_{eJ}$ | Reduced Joint load due to external load (lb) |
| $F_i$ | Preload on Bolt and Joint (lb) |
| $\Delta F_i$ | Preload loss (–lb) |
| $F_{i\,min}$ | Minimum preload (lb) |
| $F_{i\,max}$ | Maximum preload (lb) |
| $F_{j\,nom}$ | Nominal preload (lb) |
| $F_{B\,max}$ | Maximum Bolt load (lb) |
| $F_{J\,min}$ | Minimum Joint load (lb) |
| K | Spring rate (lb/in.) |
| $K_B$ | Spring rate of Bolt (lb/in.) |
| $K_J$ | Spring rate of Joint (lb/in.) |
| $K_x$ | Spring rate of Bolt part $l_x$ (lb/in.) |
| l | Length (in.) |
| $\Delta l$ | Change in length (in.) |
| $l_B$ | Length of Bolt (in.) |
| $\Delta l_B$ | Bolt elongation due to $F_i$ (in.) |
| $l_J$ | Length of Joint (in.) |
| $\Delta l_J$ | Joint compression to $F_i$ (in.) |
| $l_x$ | Length of Bolt part x (in.) |
| n | $\dfrac{\text{Length of clamped parts}}{\text{Total Joint Length}}$ |
| $\alpha$ | Tightening factor |
| $\Phi$ | Force ratio |
| $\lambda$ | Bolt and Joint elongation due to $F_e$ (in.) |
| $\sigma_B$ | Bolt stress amplitude (± psi) |